Functional determinants for ordinary differential operators

1992 J. Phys. A: Math. Gen. 25 6743

(http://iopscience.iop.org/0305-4470/25/24/025)

J. Phys. A: Math. Gen. 25 (1992) 6743–6753. Printed in the UK

Functional determinants for ordinary differential operators*

R E Gamboa Saraví‡, G L Rossini‡ and M Fuentes§

‡ Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, CC 67, (1900) La Plata, Argentina

§ Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, IL 61801, USA

Received 18 Feburary 1992, in final form 10 August 1992

Abstract. We study the analytic properties of regularized functional determinants for ordinary differential operators. To this end a new regularization scheme is introduced. Relations between different approaches are analysed and the connection between the presence of divergent traces and regularization differences is discussed.

1. Introduction

The necessity of extending the concept of the determinant of a finite square matrix to the infinite-dimensional case of elliptic operators acting on a vector bundle naturally arises from the path integral formulation of quantum physics. Several extensions—procedures that make sense in a wider range and when applied to the finite-dimensional case yield the standard determinant—have been proposed for defining the determinant. However, there is not complete agreement between the different approaches. Thus, it seems to be relevant to state clearly the relation between different results and the origin of these differences.

A widely used approach is the ζ-function determinant based on the complex powers of an elliptic operator L [1]. Since $\zeta(s) \equiv \mathrm{Tr}(L^{-s})$ admits a meromorphic extension to the whole complex plane and in particular it is analytic in the origin, one can set $\mathrm{Det}_\zeta(L) \equiv \exp(-\zeta'(0))$. It is possible to show that $\mathrm{Det}_\zeta(L-\lambda)$ is an analytic function of the parameter λ and has a zero when λ is an eigenvalue of L; the order of each zero is given by the algebraic multiplicity of the corresponding eigenvalue [2].

Another elegant method originally presented by Carleman [3] (see [4] for a modern and quantum field theory oriented presentation) is the p-determinant for $\mathcal{J}_p$ operators, i.e. operators A such that $\mathrm{Tr}(A^p)$ converges. In this approach, given an operator $A \in \mathcal{J}_p$, one constructs an associated operator $\tilde{A}$ with finite trace and defines $\mathrm{Det}_p(1+\gamma A)$ as the Fredholm determinant of $(1+\gamma\tilde{A})$ [5]. This regularization, as it occurs with the ζ-function determinant, preserves the relation between the spectrum of A and the zeros of the determinant as a function of γ (see section 3 for details).

In any case, one can see that some finite-dimensional properties are lost when extending the determinant definition, while others remain valid. Thus, choosing some

* This work was partially supported by CONICET (Argentina).

0305-4470/92/246743+11$07.50

properties to be preserved appears to be a guiding principle to define an appropriate extension. That is the basic idea of the method proposed by Coleman [6] for defining the ratio of determinants of one-dimensional Schrödinger operators. He introduced a complex parameter γ in the operators in such a way that one would expect a precise characterization of the zeros and poles of the determinant as a meromorphic function of γ. Then, from the solutions of the homogeneous initial value problems associated with the operators, a function that exhibits the same location and order of zeros and poles is built. It is this function that is taken to be the regularized determinant. This method was extended by Dreyfus and Dym [7] to the case of quotients of two ordinary differential operators of the same order with the same principal and sub-principal symbols, both acting on functions of $L^2([a,b])$.

We pursue this latter idea to extend the method to the following case: we consider a family of ordinary differential operators $L(\gamma)$ sharing only the principal symbol and acting on sections of a vector bundle $E = [a,b] \times C^r$. To definine the quotient of determinants of one of this operators, $L(\gamma)$, and a non-singular (fixed) one, $L(0)$, we introduce a function of the parameter γ. We require this function be entire and take the values one for the identity operator and zero whenever the operator $L(\gamma)$ fails to be invertible. We also require that the order of each of these zeros equal the algebraic multiplicity of the corresponding zero mode of $L(\gamma)$.

It is easy to see that the requested characterization admits the presence of a factor of the form $\exp(\gamma g(\gamma))$ multiplying the result, where g is an arbitrary entire function. Moreover, since Det_ζ and Det_p satisfy the same characterization, one should expect that the corresponding results differ for most in factors of this kind. In fact, when we compare our method with other known regularization schemes we see that a factor of this kind makes just the difference between the results. So the arbitrary nature of this factor appears to be intrinsic to the very concept of determinant for infinite-dimensional operators. Moreover, they can be seen as (finite) counterterms in the frame of renormalization theory. As we shall see in section 3, this fact arises from the presence of ill-defined traces in the loop expansion of the determinant.

Finally, let us remark that, though being restricted to one-dimensional operators acting on a finite interval, our approach could be useful even in quantum field theory in problems such that their symmetries and boundary conditions permit reduction to, for instance, a radial coordinate (see for example [8]).

The paper is organized as follows: in section 2 we state the problem to be considered and present the regularization scheme. In section 3 we compare our results with the 2-determinant and the ζ-function determinant and discuss the ambiguity between different approaches. In section 4 we illustrate with a simple example the previous results.

2. Definition of the Δ-determinant

We shall now state precisely the problem to be studied. Let us consider elliptic ordinary differential operators of order m acting on sections of a vector bundle $E = [a,b] \times C^r$ over $[a,b]$, having the following form

$$L(\gamma) = P + \gamma Q \tag{1}$$

where

$$P = p_m(x)\frac{\mathrm{d}^m}{\mathrm{d}x^m} + \cdots + p_0(x) \tag{2}$$

$$Q = q_{m-1}(x)\frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}} + \cdots + q_0(x) \tag{3}$$

and γ is a complex parameter. Ellipticity means that $\det p_m(x) \neq 0$ for each $x \in [a, b]$. The coefficients $p_i(x)$ and $q_i(x)$ are $r \times r$ complex matrix valued functions having i continuous derivatives.

Let $y(x)$ denote the sections of the vector bundle

$$y(x) = \begin{pmatrix} f_1(x) \\ \cdot \\ \cdot \\ \cdot \\ f_r(x) \end{pmatrix} \tag{4}$$

where $f_j \in C^m([a, b])$. Let also $\tilde{y}$ denote the mr-tuple built up by joining together $y(x), y'(x), \ldots, y^{(m-1)}(x)$.

In order to have elliptic boundary conditions, we impose

$$M_a \tilde{y}(a) + M_b \tilde{y}(b) = 0 \tag{5}$$

where M_a and M_b are constant complex $mr \times mr$ matrices satisfying

$$\mathrm{rank}([M_a, M_b]) = mr. \tag{6}$$

Throughout this paper we will assume that P is non-singular with these boundary conditions.

Using all this notation, we are considering the following elliptic boundary value problem

$$L(\gamma)y = 0 \tag{7}$$

$$M_a \tilde{y}(a) + M_b \tilde{y}(b) = 0. \tag{8}$$

It is our aim to construct a function $\Delta(\gamma)$ associated with the operator $L(\gamma)$ whose zeros are related to the singular L's. In order to do so, we propose the following mechanism: first, we choose the fundamental matrix $\mathbf{Y}(x, \gamma)$ of equation (7) satisfying the initial conditions

$$\mathbf{Y}(a, \gamma) = \mathbf{1}_{mr \times mr}. \tag{9}$$

Then, we define the complex matrix function

$$U(\gamma) = M_a \mathbf{Y}(a, \gamma) + M_b \mathbf{Y}(b, \gamma) \tag{10}$$

and finally, we set

$$\Delta(\gamma) = \det(U(\gamma)). \tag{11}$$

Now, let us discuss some properties of the function $\Delta(\gamma)$. They naturally will lead us to relate $\Delta(\gamma)$ with the determinant of $L(\gamma)$.

Properties.

(a) $\Delta(\gamma)$ is an entire function of the complex variable γ.

(b) $\Delta(\gamma)$ grows with finite order $\rho \leqslant 1$.

(c) $\Delta(\gamma) = 0$ if and only if $L(\gamma)$ has zero as eigenvalue.

(d) The order of the zeros of $\Delta(\gamma)$ equals the algebraic multiplicity of 0 as eigenvalue of $L(\gamma)$.

(e) $\Delta(\gamma)$ can be expanded as an infinite product of the form

$$\Delta(\gamma) = e^{\alpha\gamma+\beta} \prod_i \left(1 - \frac{\gamma}{\gamma_i}\right) e^{\gamma/\gamma_i} \tag{12}$$

where γ_i are the zeros of $\Delta(\gamma)$, repeated as many times as indicated by their order.

(f) α and β are given by the expressions

$$\alpha = \operatorname{tr}\left[U^{-1}(0)\frac{\partial}{\partial\gamma}U(\gamma)|_{\gamma=0}\right] \tag{13}$$

$$\beta = \log(\det U(0)) \tag{14}$$

where $U(\gamma)$ is the function defined in equation (10).

Proof.

(a) Since the coefficients of $L(\gamma)$ are analytic in γ, so are the solutions of equation (7) with initial values independent of γ. On the other hand, the matrices M_a and M_b do not depend on γ. Thus $\Delta(\gamma)$ is entire. □

(b) and (d) We refer the reader to lemmas 3, 6 and 7 in [7]. The extension to systems of differential equations is straightforward. □

(c) The general solution with its $m-1$ first derivatives can be written as

$$\tilde{y}(x,\gamma) = \mathbf{Y}(x,\gamma)C \tag{15}$$

where C is a mr-tuple of complex constants. Imposing the boundary conditions to it yields

$$[M_a\mathbf{Y}(a,\gamma) + M_b\mathbf{Y}(b,\gamma)]C = 0. \tag{16}$$

Thus there is a non-trivial solution if and only if $\Delta(\gamma) = 0$. □

(e) This fact follows from Hadamard's factorization theorem [9], whose hypothesis are satisfied by virtue of properties a and b. Notice that α and β are not determined by the theorem. In the case that P was singular, and thus $\Delta(0) = 0$, a factor γ^σ would also appear in equation (12), where σ would be the order of the zero of $\Delta(0)$. □

(f) β is easily evaluated by computing $\Delta(0)$ in equation (12). To evaluate α we compute $\frac{\partial}{\partial\gamma}\log\Delta(\gamma)|_{\gamma=0}$ using equation (12) and the well known result

$$\frac{\partial}{\partial\gamma}\log(\det U(\gamma)) = \operatorname{tr}\left(U^{-1}(\gamma)\frac{\partial}{\partial\gamma}U(\gamma)\right) \tag{17}$$

that holds whenever $\det U(\gamma) \neq 0$. □

Properties (a) to (e) show that $\Delta(\gamma)$ plays a similar role to that of the characteristic polynomial of $P + \gamma Q$ in finite dimension, because it respects the relation between zeros and eigenvalues.

The exponential factors inside the product symbol in equation (12) are necessary and sufficient to make convergent the infinite product. The factors outside the product symbol depend on the proposed construction. Any value can be given to the factor e^{β} just by a redefinition of the matrices M_a and M_b (notice that a left-multiplication by any non-singular matrix M does not change the boundary conditions but adds the factor Det M to the value of e^{β}). However, if we normalize the function Δ to take the value one for $\gamma = 0$ by considering the ratio $\Delta(\gamma)/\Delta(0)$, this ambiguity is removed.

The remaining factor $e^{\gamma\alpha}$ appears to be inherent to this generalization. Property (f) gives the explicit expression for this factor in terms of the proposed construction.

Following the considerations above we arrive to the natural definition:

$$\frac{\mathrm{Det}_{\Delta} L(\gamma)}{\mathrm{Det}_{\Delta} L(0)} \equiv \frac{\Delta(\gamma)}{\Delta(0)}. \tag{18}$$

Remark. For the purpose of computation it is important to notice that the matrix

$$\frac{\partial}{\partial \gamma} \mathbf{Y}(b,\gamma)|_{\gamma=0} \tag{19}$$

that appears in the expression for α can be obtained by evaluating in $x = b$ the solution of the equation

$$P_x \frac{\partial}{\partial \gamma} y(x,\gamma)|_{\gamma=0} = -Q_x y(x,0) \tag{20}$$

with homogeneous initial conditions in $x = a$.

3. Relation with other regularizations

In this section we are going to compare the Δ-determinant of $L(\gamma)$ with other regularization schemes. In doing so, we will verify that definition (18) is related to well known regularized results through finite local counterterms.

To begin with, we are going to compare our result with the p-determinant [5], which applies in general to operators of the form

$$1 + A \tag{21}$$

where $A \in \mathcal{J}_p$, i.e. $\mathrm{Tr}(A^p)$ is a well defined quantity. For them, it is possible to show that

$$1 + \tilde{A} = (1 + A) \exp\left(\sum_{j=1}^{p-1} \frac{(-1)^j}{j} A^j\right) \tag{22}$$

admits the Fredholm determinant. Then the definition of the p-determinant is as follows:

$$\mathrm{Det}_p(1 + A) = \det(1 + \tilde{A}) \tag{23}$$

where det stands for the Fredholm determinant. In terms of the eigenvalues λ_i of the operator A, one can write

$$\mathrm{Det}_p(1+A) = \prod_i (1+\lambda_i) \exp\left(\sum_{j=1}^{p-1} \frac{(-1)^j}{j} \lambda_i^j\right). \tag{24}$$

Notice that the operator A must be of negative order to belong to $\mathcal{J}_p$ for some natural p. In our case, since $L(\gamma)$ are elliptic operators of order $m > 0$, it is natural to consider the p-determinant of the operator

$$L(\gamma)L^{-1}(0) = 1 + \gamma Q P^{-1} \tag{25}$$

which should be related to the quotient $\Delta(\gamma)/\Delta(0)$.

We are going to prove in the following lemma that the minimum p for which $\mathrm{Tr}((QP^{-1})^p)$ is finite is $p = 2$.

Lemma 1. The operator QP^{-1} belongs to $\mathcal{J}_2$ (Hilbert–Schmidt class).

Proof. The kernel of P^{-1} is the Green function $P^{-1}(x,y)$ which has the following properties:

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} P^{-1}(x,y) \qquad \text{is continuous for } k \leqslant m-2 \tag{26}$$

$$\lim_{y\to x+} \frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}} P^{-1}(x,y) + p_m^{-1}(x) = \lim_{y\to x-} \frac{\mathrm{d}^{m-1}}{\mathrm{d}x^{m-1}} P^{-1}(x,y). \tag{27}$$

This implies that the kernel of QP^{-1} has at most a jump discontinuity along the diagonal. Thus $\mathrm{Tr}(QP^{-1})$ is in general not well defined. The trace of $(QP^{-1})^2$ is given by

$$\mathrm{Tr}(QP^{-1})^2 = \int_{[a,b]\times[a,b]} Q_x P^{-1}(x,y) Q_y P^{-1}(y,x)\, \mathrm{d}x\, \mathrm{d}y \tag{28}$$

and the integral is convergent because the integrand is continuous almost everywhere in $[a,b]\times[a,b]$. □

Therefore, $\mathrm{Det}_p(1+\gamma QP^{-1})$ makes sense for $p \geqslant 2$.

We are now ready to establish the relation between $\mathrm{Det}_2(L(\gamma)L^{-1}(0))$ and the quotient in equation (18). To this end we prove a lemma that relates the zeros of $\Delta(\gamma)$ to the eigenvalues of QP^{-1}:

Lemma 2. $\gamma_i \neq 0$, $-1/\gamma_i$ is an eigenvalue of QP^{-1} if and only if $L(\gamma_i)$ has zero as an eigenvalue. Moreover, they have the same algebraic multiplicity.

Proof. This follows straightforwardly from the equation

$$\gamma^{-1} L(\gamma) P^{-1} f = QP^{-1} f + \gamma^{-1} f \tag{29}$$

and the fact that $P^{-1}f$ satisfies the boundary conditions (5) for every f in $L^2([a,b])$. □

The following theorem states the relation between Det_2 and Det_Δ that we are looking for:

Theorem.

$$\mathrm{Det}_2(L(\gamma)L^{-1}(0)) = \exp\left\{-\gamma \operatorname{tr}\left[U^{-1}(0)\frac{\partial}{\partial\gamma}U(\gamma)|_{\gamma=0}\right]\right\}\frac{\mathrm{Det}_\Delta L(\gamma)}{\mathrm{Det}_\Delta L(0)}. \tag{30}$$

Proof. If we call $-1/\gamma_i$ the eigenvalues of QP^{-1}, repeated as many times as indicated by their algebraic multiplicity, from equation (24) we get

$$\mathrm{Det}_2(1+\gamma QP^{-1}) = \prod_i (1-\gamma/\gamma_i)\mathrm{e}^{\gamma/\gamma_i}. \tag{31}$$

Comparing equation (31) with the factorization in equation (12) and taking into account the result of lemma 2, we see that

$$\mathrm{Det}_2(L(\gamma)L^{-1}(0)) = \mathrm{e}^{-\alpha\gamma}\frac{\Delta(\gamma)}{\Delta(0)}. \tag{32}$$

Finally, from the definition (18) and equation (13) the theorem follows. □

The relation between $\mathrm{Det}_\Delta L(\gamma)/\mathrm{Det}_\Delta L(0)$ and $\mathrm{Det}_p(L(\gamma)L^{-1}(0))$ for $p \geqslant 3$ can be established now immediately from the following property [5]:

$$\mathrm{Det}_{p+1}(1+A) = \exp\left(\frac{(-1)^p}{p}\mathrm{Tr}(A^p)\right)\mathrm{Det}_p(1+A). \tag{33}$$

Remark. Notice that in the case where $q_{m-1}(x)$ vanishes, the Fredholm determinant of $1+\gamma QP^{-1}$ also exists [7]. Then, from equation (33), one can write

$$\det(1+\gamma QP^{-1}) = \mathrm{e}^{\gamma\mathrm{Tr}(QP^{-1})}\mathrm{Det}_2(1+\gamma QP^{-1}). \tag{34}$$

Moreover, in this case the estimate for the order of $\Delta(\gamma)$ can be reduced to $\rho \leqslant 1/2$. Then, Hadamard's theorem yields

$$\frac{\Delta(\gamma)}{\Delta(0)} = \prod_i \left(1-\frac{\gamma}{\gamma_i}\right) \tag{35}$$

which is precisely the Fredholm determinant of $1+\gamma QP^{-1}$. Taking into account equations (32) and (34), we find that

$$\mathrm{Tr}(QP^{-1}) = \alpha. \tag{36}$$

In the general case, the left-hand side of this equation does not make sense, but α is well defined from equation (13). Therefore, within our approach, it is natural to consider α as the regularized value of $\mathrm{Tr}(QP^{-1})$.

Finally, we describe the relation between the method that we propose and the ζ-function regularization. In [10], Forman was able to reduce the computation of the quotient of ζ-function regularized determinants of elliptic operators defined on manifolds with boundary to the computation of another determinant, namely the Fredholm determinant of an operator related to the boundary values of the solutions of the original operators. In the case when the manifold is a one-dimensional closed interval, and following our notation, his result reads

$$\frac{\mathrm{Det}_\zeta L(\gamma)}{\mathrm{Det}_\zeta L(0)} = \exp\left(\gamma \int_a^b \mathrm{tr}(R(x) q_{m-1}(x) p_m^{-1}(x))\,\mathrm{d}x\right) \frac{\mathrm{Det}_\Delta L(\gamma)}{\mathrm{Det}_\Delta L(0)} \tag{37}$$

where tr means trace of finite-dimensional matrices and $R(x)$ is $\frac{1}{2}$ if m is even; otherwise it is a projector onto the eigenspaces of $i^m p_m(x)$ corresponding to the eigenvalues lying on one of the half-planes limited by a minimal growth ray of P and the opposite ray.

When $q_{m-1}(x) \equiv 0$, the ζ and Δ-results agree exactly [10], as it occurs for Δ and Fredholm determinants [7]. In the general case $q_{m-1}(x) \neq 0$, the exponential factor in equation (37) can be interpreted as another regularization of $\mathrm{Tr}(QP^{-1})$ (notice that the notation in this equation is quite suggestive).

A relation between Det_p and the ζ-function regularization was found for a more general case without reference to any other method [11], and it agrees with the relation that follows from equations (30) and (37).

One can see from equations (30) and (37) that the only difference between the three different approaches we have discussed is a factor $\exp(c\gamma)$, being c γ-independent. As we mentioned in the introduction, one should have expected this fact because the three extensions respect the same relation between zeros of the determinant and zero modes of $L(\gamma)$. For vanishing $q_{m-1}(x)$, when $\mathrm{Tr}(QP^{-1})$ makes sense, all these factors dissapear and every approach gives the same answer. When $\mathrm{Tr}(QP^{-1})$ does not exist, it should be regularized, and the factors relating to the different approaches reflect the fact that different regularizations can be used for this trace. Moreover, when the manifold dimension is higher than one, even higher powers of $L(\gamma)L(0)^{-1} - 1$ have divergent traces, so that one should expect factors of the form $\exp(\gamma g(\gamma))$ relating different regularizations, where g is a polynomial in γ. When applying any one of these methods to a specific problem where divergent traces are present, one should adopt a renormalization prescription. This is done by adding counterterms to be fixed later on, one corresponding to each divergent trace. Even though different approaches for the determinant will require different counterterms to satisfy the same renormalization criteria, the final result will be independent of the chosen extension.

As we stated at the beginning of this section, equation (37) ensures that the Δ-determinant differs from the ζ-function determinant by a finite local renormalization. This fact guarantees that the former is consistent with the general principles of renormalization, in the sense that the regularized determinant corresponds to the unrenormalized one times 'local' counterterms. Indeed, as the ζ-function result is known to satisfy this general principle (see [12] for a full discussion), then our prescription satisfies this requirement too.

4. One simple example

We present in this section a simple example in which everything can be computed explicitly. It corresponds to the operators

$$P = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + m^2 \tag{38}$$

$$Q = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} \tag{39}$$

with periodic boundary conditions on the interval $[0, 2\pi]$. This means that M_0 and $M_{2\pi}$ can be chosen as the real 2×2 matrices

$$M_0 = \mathbf{1} \tag{40}$$

$$M_{2\pi} = -\mathbf{1}. \tag{41}$$

Using the fundamental solution of equation (7) satisfying the initial conditions (9) we obtain

$$\Delta(\gamma) = -4\pi \mathrm{e}^{\mathrm{i}\pi\gamma} \sin\left(\tfrac{\pi}{2}(\gamma + \sqrt{\gamma^2 - 4m^2})\right) \sin\left(\tfrac{\pi}{2}(\gamma - \sqrt{\gamma^2 - 4m^2})\right). \tag{42}$$

Thus using the definition (18) we find

$$\frac{\mathrm{Det}_\Delta L(\gamma)}{\mathrm{Det}_\Delta L(0)} = \mathrm{e}^{\mathrm{i}\pi\gamma} \frac{\sin\left(\frac{\pi}{2}(\gamma + \sqrt{\gamma^2 - 4m^2})\right) \sin\left(\frac{\pi}{2}(\gamma - \sqrt{\gamma^2 - 4m^2})\right)}{\sinh^2(m\pi)}. \tag{43}$$

To compute both the 2-determinant and the ζ-determinant, we first note that the eigenvalues of $L(\gamma)$ are

$$\lambda_k = k^2 - \gamma k + m^2 \qquad k \in \mathbb{Z}. \tag{44}$$

In terms of the eigenvalues, the 2-determinant is given by equation (24), and reads

$$\mathrm{Det}_2(L(\gamma)L^{-1}(0)) = \prod_{k=-\infty}^{\infty} \left(1 + \frac{\gamma k}{k^2 + m^2}\right) \exp\left(-\frac{\gamma k}{k^2 + m^2}\right) \tag{45}$$

where the product can be evaluated using the formula

$$\sin z = z \prod_{j=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 j^2}\right). \tag{46}$$

The result is

$$\mathrm{Det}_2(L(\gamma)L^{-1}(0)) = \frac{\sin\left(\frac{\pi}{2}(\gamma + \sqrt{\gamma^2 - 4m^2})\right) \sin\left(\frac{\pi}{2}(\gamma - \sqrt{\gamma^2 - 4m^2})\right)}{\sinh^2(m\pi)}. \tag{47}$$

The difference between the right-hand sides of equation (43) and equation (47) is the factor $\exp(i\pi\gamma)$. It agrees with its expected value (cf equation (30))

$$\operatorname{tr}\left[U^{-1}(0)\frac{\partial}{\partial\gamma}U(\gamma)|_{\gamma=0}\right] = i\pi. \tag{48}$$

The ζ-determinant is given by the expression

$$\mathrm{Det}_\zeta(L(\gamma)) = \exp\left(-\frac{d}{ds}(\zeta(s)|_{s=0})\right) \tag{49}$$

where

$$\zeta(s) = \mathrm{Tr}(L^{-s}) \tag{50}$$

$$= \sum_{k=-\infty}^{\infty}\frac{1}{(\lambda_k)^s} \tag{51}$$

for $\mathrm{Re}(s) > \frac{1}{2}$. Using the numeric Riemman ζ-function, $\zeta(s)$ can be analitically extended to the whole complex plane as a meromorphic function. Following these steps we find

$$\mathrm{Det}_\zeta L(\gamma) = 4\sin\left(\tfrac{\pi}{2}(\gamma+\sqrt{\gamma^2-4m^2})\right)\sin\left(\tfrac{\pi}{2}(\gamma-\sqrt{\gamma^2-4m^2})\right). \tag{52}$$

To facilitate the comparison we write down the ratio

$$\frac{\mathrm{Det}_\zeta L(\gamma)}{\mathrm{Det}_\zeta L(0)} = \frac{\sin\left(\frac{\pi}{2}(\gamma+\sqrt{\gamma^2-4m^2})\right)\sin\left(\frac{\pi}{2}(\gamma-\sqrt{\gamma^2-4m^2})\right)}{\sinh^2(m\pi)}. \tag{53}$$

The exponential factor that relates the results (43) and (53) is correctly given by equation (37)

$$\gamma\int_a^b \operatorname{tr}\left(\tfrac{1}{2}q_{m-1}(x)p_m^{-1}(x)\right)\,dx = -i\pi\gamma. \tag{54}$$

Acknowledgments

The authors would like to thank J Solomín, M A Muschietti and especially O Barraza for helpful discussions about this work.

References

[1] Seeley R T 1967 Complex powers of an elliptic operator *Am. Math. Soc. Proc. Sym. Pure Math.* **10** 288–307

[2] Gamboa-Saraví R E, Muschietti M A and Solomín J E 1984 On the regularized determinant for non-invertible elliptic operators *Commun. Math. Phys.* **93** 407–15

[3] Carleman T 1921 Zur Theorie der Linearen Integralgleichungen *Math. Zeit.* **9** 196–217

[4] Seiler E 1975 Schwinger functions for the Yukawa model in two space dimensions with space–time cutoff *Commun. Math. Phys.* **42** 163–82

Seiler E and Simon B 1975 On finite mass renormalization in the two-dimensional Yukawa model *J. Math. Phys.* **16** 2289–93

[5] Simon B 1979 *Trace Ideals and Their Applications* (Cambridge: Cambridge University Press)

[6] Coleman S 1977 *The Uses of Instantons* Erice Lectures, ed A Zichichi

[7] Dreyfus T and Dym H 1978 Product formulas for the eigenvalues of a class of boundary value problems *Duke Math. J.* **45** 15–37

[8] Yang C N 1987 Generalization of Sturm-Liouville theory to a system of ordinary differential equations with Dirac type spectrum *Commun. Math. Phys.* **112** 205–16

[9] Markushevich A I 1965 *Theory of Functions of a Complex Variable* (New Jersey: Prentice-Hall)

[10] Forman R 1987 Functional determinants and geometry *Invent. Math.* **88** 447–93

[11] Barraza O 1992 in press

[12] Seiler E 1982 *Gauge Theories: Fundamental Interactions and Rigorous Results* ed P Dita *et al* (Boston: Birkhauser)